Stochastic wave equations with values in Riemannian manifolds

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1. Introduction

Because of their numerous applications in natural sciences the wave equations form a classical research field both in mathematics and physics. They describe propagation of all kinds of waves in space or in some media. One can well imagine acoustic waves (sound), propagation of light, electromagnetic waves (radio), ocean waves, seismic waves (earthquake), ultrasonic waves (detection of flaws in materials), vibrations of strings, airplane wings, membranes or elastic bodies etc.

The development of modern theoretical physics has widely extended the scope of applications of wave equations to fields such as harmonic gauges in general relativity, non-linear σ-models in particle systems, electro-vacuum Einstein equations, the non-abelian gauge theories (which serve as a description of elementary particle processes) or non-linear optics, see e.g. Misner [23], Forger [12], Gu [17], Choquet-Bruhat & Christodoulou [7], Choquet-Bruhat & Segal [8], Eardley & Moncrief [10], Ginibre & Velo [15], Glassey & Strauss [16]. In all these models wave equations appear in a Riemannian geometry setting, the so called geometric wave equations, where waves do not travel in an Euclidean space but in a Riemannian manifold like a surface, a Lie group, a warp product or a homogeneous space. These models pose interesting and challenging problems from the point of view of non-linear hyperbolic PDEs and thus belong to one of the mainstreams in the current mathematical research with the aim to justify or modify the physical theories. For example, the field theory deals with maps between a Minkowski $d+1$-dimensional space-time manifold Σ and Riemannian manifold M, see e.g. [14]. From the mathematical point of view, the wave maps, i.e. the solutions of GWEs, are the maps between Σ and M that are stationary points of the action functional $\mathcal{A}(u) = \int\int(|u_x|^2 - |u_t|^2)/2\,dxdt$. Therefore the wave maps could be viewed as semi-classical limits of field theories and, in local coordinates, they satisfy the following non-linear wave equation

$$\partial_{tt}u^k = \Delta u^k + \Gamma^k_{ij}(u)\partial_\alpha u^i \partial^\alpha u^j, \qquad u(0) = u_0, \quad u_t(0) = v_0. \tag{1.1}$$

We intend to run this part of our project in parallel with general research on deterministic theory of geometric wave equations. We will concentrate on the study of stochastic perturbations of wave equations in Riemannian manifolds - an unexplored area. This a continuation of our previous project that resulted in a paper [3], as far as we are aware the only one on the subject, in which we have suggested two natural formulations of the stochastic GWE, proved their equivalence, as well as the existence and uniqueness of a strong global solution. The results are valid for every target Riemannian manifold but only for the $(1+1)$ Minkowski space.

The aim of the current paper is to present results from [3] in a more accessible way to specialists in Stochastic PDEs who are not specialist in differential geometry. Hence special care is taken in logical introducing all necessary geometric background material. But we also provide different and often clearer proofs and statements than those presented in [3].

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2. Necessary background on differential geometry

We assume that the reader is familiar with notions of a differentiable (and riemannian) manifold, tangent space, vector field. From now we assume that M is is a compact riemannian manifold. By T_pM, $p \in M$, we will denote the tangent space to M at p, and by $\pi : TM \to M$ we will denote the tangent vector bundle. The space of all smooth vector fields on M, i.e. sections of π, will be denoted by $\mathfrak{X}(M)$. The space of all smooth $\mathbb{R}$-valued functions on M will be denoted by $\mathfrak{F}(M)$. If $I \subset \mathbb{R}$ is an open interval and $\gamma : I \to M$ is a smooth map, then by $\partial_t\gamma(t) \in T_{\gamma(t)}M$, or simply by $\gamma'(t)$, we will denote the tangent vector to γ at $t \in I$.

One should recall an alternative equivalent definition of a vector field, namely a vector field on M is a smooth $\mathbb{R}$-linear map $X : \mathfrak{F}(M) \to \mathfrak{F}(M)$ such that

(D0) $X(fg) = X(f)g + fX(g)$, for all $f, g \in \mathfrak{F}(M)$.

We will exchangeably use these two different approaches to a vector field.

We will list here necessary definitions and theorems that we will later on use in the paper.

Definition 2.1. *[26, Definition 3.9] A connection on a smooth manifold M is a function $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto \nabla_X Y \in \mathfrak{X}(M)$ such that*

(D1) for each $Y \in \mathfrak{X}(M)$, the map $\mathfrak{X}(M) \ni X \mapsto \nabla_X Y \in \mathfrak{X}(M)$ is $\mathfrak{F}(M)$-linear,

(D2) for each $X \in \mathfrak{X}(M)$, the map $\mathfrak{X}(M) \ni Y \mapsto \nabla_X Y \in \mathfrak{X}(M)$ is $\mathbb{R}$-linear,

(D3) for all $X, Y \in \mathfrak{X}(M)$ and $f \in \mathfrak{F}(M)$, $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$.

$\nabla_X Y$ is called the covariant derivative of Y with respect to X for the connection ∇.

As is explained in [26], the axiom (D1) in view of [26, Proposition 2.2], for any $Y \in \mathfrak{X}(M)$ and each $p \in M$ and each individual tangent vector $u \in T_p(M)$, a tangent vector $\nabla_u Y \in T_p(M)$ is well defined. To be precise, $\nabla_u Y = \nabla_X Y(p)$, where $X \in \mathfrak{X}(M)$ satisfies $X(p) = u$.

Let us recall the following fundamental result due to Levi-Civita.

Theorem 2.1. *[26, Theorem 3.11] If (M, g) is a riemannian manifold then there exists a unique connection ∇ on M, called the Levi-Civita connection such that for all $X, Y, Z \in \mathfrak{X}(M)$,*

(D4) $[X, Y] = \nabla_X Y - \nabla_Y X$, (D5) $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$, where $\langle Y, Z\rangle : M \ni p \mapsto g_p(Y(p), Z(p)) \in \mathbb{R}$.

Let us also recall the following result about differentiating along a curve. For a smooth map $\gamma : I \to M$ we will denote by $\mathfrak{X}(\gamma)$ the space of all smooth vector fields on γ and if $V \in \mathfrak{X}(M)$ then $(V_\gamma)(t) = V(\gamma(t))$, $t \in I$. By $\mathfrak{F}(I)$ we will denote the space $C^\infty(I, \mathbb{R})$.

Proposition 2.1. *[26, Proposition 3.18] or [20, Theorem 9.1] Assume that ∇ is the Levi-Civita connection on a riemannian manifold M. If $I \subset \mathbb{R}$ is an open interval and $\gamma : I \to M$ is a smooth map, then there exists a unique linear map $' : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$ such that*

(i2) $(hZ)' = (\frac{dh}{dt})Z + hZ'$, for all $h \in \mathfrak{F}(I)$, $Z \in \mathfrak{X}(\gamma)$,

(i3) $(V_\gamma)'(t) = \nabla_{\partial_t\gamma(t)}(V)$, $t \in I$ *for all* $V \in \mathfrak{X}(M)$.

Furthermore, (i4) $\frac{d}{dt}\langle Z_1, Z_2\rangle = \langle Z_1', Z_2\rangle + \langle Z_1, Z_2'\rangle$, *for all* $Z_1, Z_2 \in \mathfrak{X}(\gamma)$.

We will denote $Z'(t)$ by $\nabla_{\partial_t\gamma(t)}(Z)(t)$. In particular, if $Z(t) = \partial_t\gamma(t)$, $t \in I$, is the velocity field of γ, then $\nabla_{\partial_t\gamma(t)}(\partial_t\gamma)(t)$ is called the *acceleration* of the curve γ at $t \in I$ and will be denoted in this paper by $\mathbf{D}_t\partial_t\gamma(t)$. Please note that the time variable will sometimes be denoted by s and also that the same construction works for a space variable x.

Example 2.1 - The euclidean space $\overline{M} = \mathbb{R}^d$ equipped with a trivial metric tensor g is a riemannian manifold. For each $p \in \mathbb{R}^d$, the tangent space $T_p\mathbb{R}^d$ is naturally isometrically isomorphic to $\mathbb{R}^d$. Hence a vector field on X on $\mathbb{R}^d$ is simply a function $X : \mathbb{R}^d \to \mathbb{R}^d$. A function $\overline{\nabla} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ defined by $(\overline{\nabla}_X Y)(p) := (d_pY)(X(p))$ is the corresponding Levi-Civita connection and is called the *natural* connection on $\mathbb{R}^d$. In particular, the acceleration of a smooth curve $\gamma : I \to \mathbb{R}^d$ with respect to the *natural* connection on $\overline{M} = \mathbb{R}^d$ satisfies

$$\overline{\nabla}_{\partial_t\gamma(t)}(\partial_t\gamma)(t) = \partial_t^2\gamma(t) = \ddot{\gamma}(t), \; t \in I. \tag{2.1}$$

We have now the following fundamental geometric version of integration by parts formula.

Corollary 2.1. *If* $\varphi : J \to \mathbb{R}$ *is a smooth function with compact support and* $u : J \to M$ *is smmoth, then for every* $Z \in \mathfrak{X}(M)$,

$$\begin{aligned} &-\int_J \frac{d\varphi}{dx}(x)\langle \partial_x u(x), Z(u(x))\rangle\, dx \\ &= \int_J \varphi(x)\langle \mathbf{D}_x\partial_x u(x), Z(u(x))\rangle\, dx + \int_J \varphi(x)\langle \partial_x u(x), \nabla_{\partial_x u(x)} Z\rangle\, dx, \end{aligned} \tag{2.2}$$

where e.g. $\langle \mathbf{D}_x\partial_x u(x), Z(u(x))\rangle = \langle \mathbf{D}_x\partial_x u(x), Z(u(x))\rangle_{T_{u(x)}M}$.

Example 2.2 - Let us fix $p, q \in M$ and consider a set $\mathcal{M}_{p,q}$ of all continuous functions $\gamma : [0,1] \to M$ such that $\gamma(0) = p$, $\gamma(1) = q$, γ is absolutely continuous and $E(\gamma) = \int_0^1 |\partial_t\gamma(t)|^2\, dt$ is finite, where $|\partial_t\gamma(t)|^2 = g_{\gamma(t)}(\partial_t\gamma(t), \partial_t\gamma(t))$, $t \in [0,1]$. Then, it is known that $\mathcal{M}_{p,q}$ is a Hilbert manifold and that E is a smooth map from $\mathcal{M}_{p,q}$ to $\mathbb{R}$. Using the integration by parts formula (2.2) one can prove that if $\gamma \in \mathcal{M}_{p,q}$ is a stationary point of E, then $\mathbf{D}_t\partial_t u(t) = 0$ for all $t \in (0,1)$.

According to the celebrated Nash embedding Theorem, see [24], since M is a riemannian manifold, there exist $d \in \mathbb{N}$ and an **isometric** embedding $i : M \hookrightarrow \mathbb{R}^d$, where $\mathbb{R}^d = \overline{M}$ is the euclidean space. Hence M can be identified with its image in $\mathbb{R}^d$. In this case, i.e. when M is a riemannian submanifold of $\mathbb{R}^d$, one introduces the second fundamental form A of the submanifold M of $\mathbb{R}^d$ in such a way that $A_p : T_pM \times T_pM \to N_pM$, $p \in M$. Here N_p is the normal space to $p \in M$ with respect to to the metric in $\mathbb{R}^d$. In particular, for each $p \in M$, $\mathbb{R}^d = T_pM \oplus N_pM$. Then, see [26, p. 100], if X be a vector field on M, $p \in M$ and $\widetilde{X}$ is a smooth $\mathbb{R}^d$-valued extension of X to an $\mathbb{R}^d$-open neighbourhood V of p, then

$$(d_p\widetilde{X})(\eta) = \nabla_\eta X \oplus A_p(X(p), \eta), \qquad \eta \in T_pM, \tag{2.3}$$

where $d_q\widetilde{X} \in \mathsf{Hom}(\mathbb{R}^d, \mathbb{R}^d)$ is the Fréchet derivative of $\widetilde{X} : V \to \mathbb{R}^d$ at $q \in V$, ∇ is the Levi-Civita connection on M.

Moreover, if $\gamma : I \to M$ is a smooth curve and $X \in \mathfrak{X}_M(\gamma)$, $\bar{\gamma} = i \circ \gamma : I \to \overline{M}$ and $\bar{X} := i_*(X) \in \mathfrak{X}_{\overline{M}}(\bar{\gamma})$ is defined by $\bar{X}(t) := (d_{\gamma(t)} i)\big(X(\gamma(t))\big)$, $t \in I$, then, see [26, Proposition 4.8], for all $t \in I$,

$$\begin{aligned} \bar{\nabla}_{\partial_t \bar{\gamma}(t)} \bar{X} &= \nabla_{\partial_t \gamma(t)} X \oplus A_{\gamma(t)}(X(\gamma(t)), \partial_t\gamma(t)), \\ \bar{X}^{\cdot}(t) &= X'(t) + \nabla_{\partial_t\gamma(t)} X \oplus A_{\gamma(t)}(X(\gamma(t)), \partial_t\gamma(t)), \end{aligned} \tag{2.4}$$

where $' : \mathfrak{X}_{\overline{M}} \to \mathfrak{X}_{\overline{M}}$ and $^{\cdot} : \mathfrak{X}_M \to \mathfrak{X}_M$ are the linear maps whose existence is guaranteed by Proposition 2.1 and $\overline{\nabla}$ is the natural connection on $\overline{M} = \mathbb{R}^d$ as in Example 2.1.

In particular, but see also [26, Corollary 4.8], by applying equality (2.1) and Example 2.1 we infer that for any smooth curve $\gamma : I \to M$, where $I \subset \mathbb{R}$,

$$\left\{ \begin{aligned} \mathbf{D}_t\partial_t\gamma(t) &= \partial_{tt}\gamma(t) - A_{\gamma(t)}(\partial_t\gamma(t), \partial_t\gamma(t)), \\ A_{\gamma(t)}(\partial_t\gamma(t), \partial_t\gamma(t)) &\perp \partial_{tt}\gamma(t) - A_{\gamma(t)}(\partial_t\gamma(t), \partial_t\gamma(t)), \end{aligned} \right. \quad t \in I. \tag{2.5}$$

Note the following fundamental consequence of the above properties.

$$\langle \partial_{tt}\gamma(t) - A_{\gamma(t)}(\partial_t\gamma(t), \partial_t\gamma(t)), \partial_{tt}\gamma(t)\rangle = |\partial_{tt}\gamma(t) - A_{\gamma(t)}(\partial_t\gamma(t))|^2, \; t \in I, \tag{2.6}$$

where $|\cdot|$ and $\langle\cdot,\cdot\rangle$ are the norm and the inner product in $\mathbb{R}^d$.

We finish with the following result useful later on. By using a partition of unity we can find a finite system of vector fields $Z_1,\dots,Z_k$ on M such that

$$\xi = \sum_{i=1}^{k}\langle \xi, Z_i(p)\rangle Z_i(p), \qquad p\in M, \xi\in T_pM, \tag{2.7}$$

where $\langle \xi, Z_i(p)\rangle = g_p(\xi, Z_i(p))$ and, if M is isometrically embedded in $\mathbb{R}^d$, $\langle \xi, Z_i(p)\rangle = \langle \xi, Z_i(p)\rangle_{\mathbb{R}^d}$.

In the latter case let us denote the Euclidean components of the vector Z_i by $Z_i^1,\dots,Z_i^d$, i.e. $Z_i=\sum_{j=1}^d Z_i^j\, e_j$, where $(e_j)_{j=1}^d$ is the natural ONB of $\mathbb{R}^d$. Then we have the following simple but important consequence of (2.7).

Proposition 2.2. *If the vector fields Z_i, $i=1,\cdots,k$ satisfy* (2.7) *then*

$$\sum_{j=1}^{d}\sum_{i=1}^{k}\langle \xi, \nabla_\xi (Z_i^j Z_i)\rangle e_j = A_p(\xi,\xi), \quad p\in M, \quad \xi\in T_pM. \tag{2.8}$$

From properties of the Levi-Civita connection we get the following result.

Lemma 2.1. *If the vector fields Z_i satisfy* (2.7) *then*

$$\sum_{i=1}^{k}\langle \eta, \nabla_\xi Z_i\rangle Z_i + \sum_{i=1}^{k}\langle \eta, Z_i\rangle \nabla_\xi Z_i = 0, \qquad \eta,\xi\in T_pM, \quad p\in M. \tag{2.9}$$

Proof – Let us first show that for every $J\in\mathfrak{X}(M)$

$$0=\sum_{i=1}^{k}\langle J, \nabla_\xi Z_i\rangle Z_i + \sum_{i=1}^{k}\langle J, Z_i\rangle \nabla_\xi Z_i,\ p\in M, \xi\in T_pM. \tag{2.10}$$

By (2.7) we have $J(p)=\sum_{i=1}^k\langle J(p), Z_i(p)\rangle Z_i(p)$, $p\in M$. Hence by properties (D2), (D3) in [26, Def 3.9] we have

$$\nabla_\xi J = \sum_{i=1}^{k}\nabla_\xi(\langle J, Z_i\rangle Z_i) = \sum_{i=1}^{k}\xi(\langle J, Z_i\rangle) Z_i + \sum_{i=1}^{k}\langle J, Z_i\rangle\nabla_\xi Z_i.$$

By property (D5) in [26, Thm 3.11] we have for each i,

$$\xi(\langle J, Z_i\rangle) = \langle \nabla_\xi J, Z_i\rangle Z_i + \langle J, \nabla_\xi Z_i\rangle Z_i.$$

Since by (2.7) $\nabla_\xi J$ is equal to $\sum_{i=1}^k\langle\nabla_\xi J, Z_i\rangle Z_i$, the result follows.
The equality (2.9) follows from (2.10) by choosing J such that $J(p)=\eta$. □

Proof of Proposition 2.2– Let us next extend the vector fields Z_i to a small $\mathbb{R}^d$-open neighbourhood V of $p \in M$ and denote the extensions by $\tilde{Z}_i$. Then, $\big(\xi(Z_i^1), \cdots, \xi(Z_i^n)\big) = \big(\xi(\tilde{Z}_i^1), \cdots, \xi(\tilde{Z}_i^n)\big) = \big((d_p\tilde{Z}_i^1)\xi, \cdots, (d_p\tilde{Z}_i^n)\xi\big) = (d_p\tilde{Z})\xi$ and so by (2.3),

$$\big(\xi(Z_i^1), \cdots, \xi(Z_i^n)\big) = \nabla_\xi Z_i + A_p(\xi, Z_i(p)), \qquad \xi \in T_pM, \quad p \in M. \tag{2.11}$$

Let us denote $A = (A^1, \ldots, A^n)$ and fix $j \in \{1, \cdots, d\}$. Taking a scalar product of both sides of (2.11) with $(\langle \xi, Z_i(p)\rangle)_{i=1}^k$, we obtain, in view of (2.9),

$$\sum_{i=1}^k \xi(Z_i^j)\langle \xi, Z_i\rangle + \sum_{i=1}^k Z_i^j \langle \xi, \nabla_\xi Z_i\rangle = A_p^j(\xi, \xi).$$

On the other hand by (D3) in [26, Def 3.9], LHS of the above equality is equal to $\sum_{i=1}^k \langle \xi, \nabla_\xi (Z_i^j Z_i)\rangle$. Thus, by multiplying the resulting equality by a vector e_j and then summing over $j = 1, \cdots, d$ we get identity (2.8) and so we conclude the proof of Proposition 2.2. □

3. The SWE on $\mathbf{R}^{1+1}$ space-time

In this section we assume that M is a compact riemannian manifold. We consider the following one-dimensional stochastic wave equation

$$\mathbf{D}_t\partial_t u = \mathbf{D}_x\partial_x u + Y_u(\partial_t u, \partial_x u)\,\dot{W}. \tag{3.1}$$

where Y is a fiber-preserving C^1-class map from $TM \times TM$ to TM, where $TM \times TM$ is the cartesian product of the tangent vector bundle TM by itself. We assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a probability space, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration on it such that $\mathcal{F}_0$ contains all $\mathbb{P}$-negligible sets, and $W = (W(t))_{t\geq 0}$ is a spatially homogeneous $\mathbb{F}$-Wiener process on $\mathbb{R}$ whose a spectral measure μ satisfies the following assumptions.

Notation 3.1. *If $\mathcal{O} \subset \mathbb{R}^d$ is an open set and $k \in \mathbb{N}$ then $H^k(\mathcal{O}, \mathbb{R}^d)$ denotes standard Sobolev space of order k. We put $\mathcal{H}^k = H^{k+1}(\mathbb{R};\mathbb{R}^d) \oplus H^k(\mathbb{R};\mathbb{R}^d)$, $\mathcal{H} := \mathcal{H}^1$ and, for $R > 0$, $\mathcal{H}_R^k = H^{k+1}((-R,R);\mathbb{R}^d) \oplus H^k((-R,R);\mathbb{R}^d)$ and $\mathcal{H}_R := \mathcal{H}_R^1$. If K be a manifold and $k \in \mathbb{N}$ then a function $f : \mathbb{R} \to K$ is*

said to belong to $H^k_{loc}(\mathbb{R},K)$ *iff* $\psi(\theta\circ f)\in H^k(\mathbb{R})$ *for all* $\theta\in C^\infty(K,\mathbb{R})$ *and* $\psi\in C_0^\infty(\mathbb{R},\mathbb{R})$. *We also equip* $H^k_{loc}(\mathbb{R},K)$ *with the topology induced by the mappings* $H^k_{loc}(\mathbb{R},K)\ni f\mapsto\theta\circ f\in H^k(\mathbb{R})$, $\theta\in C^\infty(K,\mathbb{R})$, $\psi\in C_0^\infty(\mathbb{R},\mathbb{R})$.
Let us denote by $\mathfrak{S}=(S_t)_{t\in\mathbb{R}}$ *the* C_0 *group of bounded linear operators on the space* $\mathcal{H}^0$, *generated by an operator* $\mathcal{G}:=\begin{bmatrix}0 & I\\ \Delta & 0\end{bmatrix}$ *with* $D(\mathcal{G})=\mathcal{H}^2$. *The restriction of the* $\mathfrak{S}$ *to any of the spaces* $\mathcal{H}^k$, $k\in\mathbb{N}$ *is also a* C_0 *group on* $\mathcal{H}^k$. *The semigroup* $\mathfrak{S}$ *acts via well know formulae, see e.g. Section 4 in [27]. Moreover,* $\mathfrak{S}$ *extends to a* C_0 *group of linear continuous operators on the space* $\mathcal{H}^k_{\rm loc}$, *and*

$$\|S_tz\|_{\mathcal{H}_{T-|t|}}\le e^{\frac{|t|}{2}}\|z\|_{\mathcal{H}_T},\ z\in\mathcal{H}_{loc}. \tag{3.2}$$

Assumptions 3.1. *The reproducing kernel Hilbert space* H_μ *of the law of* $W(1)$ *is contained in* $H^1_{loc}(\mathbb{R})$ *and for each* $R>0$ *there exists a constants* $c_R>0$, *such that for every* $j\in\{0,1\}$ *and* $g\in H^j(\mathbb{R})$,

$$\|\xi\mapsto g\cdot\xi\|_{\mathcal{T}_2(H_\mu,H^j(-R,R))}\le c_R|g|_{H^j(-R,R)}. \tag{3.3}$$

For the deterministic version of our problem one can consult Shatah & Struwe [27]. Contrary to the deterministic case, solutions to the stochastic wave equation cannot be easily defined using local coordinates and so we suggest two possible approaches to this issue. Firstly, we propose the following definition of *intrinsic solution* which however is preceded by the following general assumptions.

Assumptions 3.2. *Functions* u_0, v_0 *are* $\mathcal{F}_0$*-measurable random variables with values in* $H^2_{loc}(\mathbb{R},M)$ *and* $H^1_{loc}(\mathbb{R},TM)$ *respectively such that* $v_0(x,\omega)\in T_{u_0(x,\omega)}M$ *for all* $\omega\in\Omega$ *and* $x\in\mathbb{R}$.

Definition 3.1. *Assume that Assumptions 3.2 are satisfied. A process* $u:\mathbb{R}_+\times\mathbb{R}\times\Omega\to M$ *is called an* **intrinsic solution** *of problem* (3.1) *with initial data (3.4), where*

$$u(0,\cdot)=u_0,\qquad \partial_tu(t,\cdot)_{|t=0}=v_0, \tag{3.4}$$

provided the following six conditions are satisfied.

(i) $u(t,x,\cdot)$ *is* $\mathcal{F}_t$*-measurable for every* $x \in \mathbb{R}$ *and every* $t \geq 0$,

(ii) $u(\cdot,\cdot,\omega)$ *belongs to* $C^1(\mathbb{R}_+ \times \mathbb{R}; M)$ *for every* $\omega \in \Omega$,

(iii) $\mathbb{R}^+ \ni t \mapsto u(t,\cdot,\omega) \in H^2_{loc}(\mathbb{R}, M)$ *is continuous for every* $\omega \in \Omega$,

(iv) $\mathbb{R}^+ \ni t \mapsto u(t,\cdot,\omega) \in H^1_{loc}(\mathbb{R}, \mathbb{R}^d)$ *is continuously differentiable for every* $\omega \in \Omega$,

(v) $u(0,x,\omega) = u_0(x,\omega)$, $\partial_t u(0,x,\omega) = v_0(x,\omega)$ *for every* $x \in \mathbb{R}$, $\mathbb{P}$*-a.s.,*

(vi) and for every vector field X *on* M*, and every* $t \geq 0$ *and* $R > 0$*, the following equality holds in* $L^2(-R,R)$*,* $\mathbb{P}$ *almost surely*

$$\langle \partial_t u(t), X(u(t)) \rangle_{T_{u(t)}M} = \langle v_0, X(u_0) \rangle_{T_{u(t)}M} + \int_0^t \langle \mathbf{D}_x \partial_x u(s), X(u(s)) \rangle_{T_{u(s)}M} \, ds$$
$$+ \int_0^t \langle \partial_t u(s), \nabla_{\partial_s u(s)} X \rangle_{T_{u(s)}M} \, ds + \int_0^t \langle X(u(s)), Y_{u(s)}(\partial_s u(s), \partial_x u(s)) \, dW(s) \rangle_{T_{u(s)}M}.$$

Remark 3.1 - In view of (2.5) the following are two equivalent differential equation formulations of the integral equation (3.5). We will not dwell upon this equivalence in this paper.

$$(3.5) \quad \langle \mathbf{D}_t \partial_t u, X(u) \rangle_{T_u M} = \langle \mathbf{D}_x \partial_x u, X(u) \rangle_{T_u M} + \langle Y_u(\partial_t u, \partial_x u) \, \dot{W}, X(u) \rangle_{T_u M},$$

for every vector field X on M.

$$(3.6) \quad \begin{cases} \partial_t \langle \partial_t u, X(u) \rangle_{T_u M} = \langle \mathbf{D}_x \partial_x u, X(u) \rangle_{T_u M} + \langle \partial_t u, \nabla_{\partial_t u} X \rangle_{T_u M} + \\ \qquad\qquad + \langle Y_u(\partial_t u, \partial_x u) \, \dot{W}, X(u) \rangle_{T_u M} \end{cases}$$

for every vector field X on M.

Now we propose the following definition of an *extrinsic solution.*

Definition 3.2. *Assume that Assumptions 3.2 are satisfied. A process* $u : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to M$ *is called an* extrinsic solution *to problem* (3.1) *with initial data (3.4), provided the five conditions (i)-(v) from Definition 3.1 are satisfied and instead of condition (vi) the following one holds* *(vii) for all $t \geq 0$ and $R > 0$ the following equality holds in $L^2((-R,R),\mathbb{R}^d)$, $\mathbb{P}$ almost surely,*

$$\begin{aligned}
\partial_t u(t) &= v_0 + \int_0^t \left[\partial_{xx}u(s) - A_{u(s)}(\partial_x u(s), \partial_x u(s)) + A_{u(s)}(\partial_s u(s), \partial_s u(s))\right] ds \\
(3.7)\ + & \int_0^t Y_{u(s)}(\partial_s u(s), \partial_x u(s))\, dW(s).
\end{aligned}$$

Remark 3.2 - Let us observe that equation (3.7) is a mild form of the following equation

$$\partial_{tt}u = \partial_{xx}u - A_u(u_x,u_x) + A_u(u_t,u_t) + Y_u(\partial_t u, \partial_x u)\,\dot{W} \tag{3.8}$$

We begin our exposition with the following result that shows that in fact our two definitions of a solution are equivalent.

Theorem 3.1. *Assume that the spatially homogeneous $\mathbb{F}$-Wiener process $W = (W(t))_{t\geq 0}$ on $\mathbb{R}$ satisfying Assumption 3.1. Suppose also that M is a compact submanifold of $\mathbb{R}$ as in Definition 3.2. Assume that Assumptions 3.2 are satisfied. Then a process $u : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to M$ is an intrinsic solution to problem (3.4) if and only if it is an extrinsic solution to the same problem.*

The above result justifies the use on a notion **solution** to problem (3.4).

Next we formulate the main result of this part of the paper.

Theorem 3.2. *Assume that the vector bundles homomorphism $Y : TM \times TM \to TM$ is of C^1 class such that both Y and TY are uniformly of linear growth on the fibers, i.e. there exists $C > 0$ such that for all $p \in M$, $\xi, \eta \in T_pM$ and $\xi_i, \eta_i \in T_pM$, $i = 1,2$,*

$$\begin{aligned}
(3.9)\qquad |Y_p(\xi,\eta)|_{T_pM} &\leq C(1 + |\xi|_{T_pM} + |\eta|_{T_pM}), \\
(3.10)\quad |d_{(p,\xi_1,\xi_2)}Y(\eta,\eta_1,\eta_2)| &\leq C\Big[\big(1+|\xi_1|+|\xi_2|\big)|\eta| + |\eta_1| + |\eta_2|\Big].
\end{aligned}$$

Then there exists an $\mathbb{F}$-adapted process $u = \big(u(t)\big)_{t\geq 0}$ such that u is a solution to problem (refequa1) with the initial data (3.4).
Moreover, suppose that $u = \big(u(t)\big)_{t\geq 0}$ and $\bar{u} = \big(\bar{u}(t)\big)_{t\geq 0}$ are two $\mathbb{F}$-adapted process such that for some $R > T > 0$ the following conditions are satisfied, $\mathbb{P}$-almost surely,

(•1) The paths of both u and $\bar{u}$ are $H^2((-R,R);\mathbb{R}^d)$-valued continuous;

(•2) The paths of both u and $\bar{u}$ are $H^1((-R,R);\mathbb{R}^d)$-valued continuously differentiable;

(•3) Both $u(t,x,\omega)$ and $\bar{u}(t,x,\omega)$ belong to M for all $x \in (-R,R)$, $t \in [0,T)$,

(•4) $u(0,x,\omega) = \bar{u}(0,x,\omega) = u_0(x,\omega)$ for all $x \in (-R,R)$,

(•5) $\partial_t u(0,x,\omega) = v_0(x,\omega)$ for all $x \in (-R,R)$,

(•6) Both u and $\bar{u}$ satisfy the equation (3.7) in $L^2((-R,R);\mathbb{R}^d)$ for all $t \in [0,R)$.

Then $\bar{u}(t,x,\omega) = u(t,x,\omega)$ for all $x \in (-(R-t),R-t)$ and $t \in [0,T)$, $\mathbb{P}$-almost surely.

The following Lemma about energy inequalities, which can be seen as a stochastic analogue of energy estimates for solutions of wave equations, for solutions of linear wave equations with an additive noise plays a fundamental rôle in our proofs.

Proposition 3.1. *Assume that $T > 0$, $k \in \mathbb{N}$ and U is a Hilbert space. Let W be a U-cylindrical $\mathbb{F}$-Wiener process. Let f and g be progressively measurable processes with values in $H^k(\mathbb{R};\mathbb{R}^d)$ and $\mathcal{L}_2(U, H^k(\mathbb{R};\mathbb{R}^d))$ respectively such that*

$$\int_0^T \left\{ |f(s)|_{H^k(\mathbb{R};\mathbb{R}^d)} + \|g(s)\|^2_{\mathcal{L}_2(U,H^k(\mathbb{R};\mathbb{R}^d))} \right\} ds < \infty \tag{3.11}$$

almost surely. Let $z_0 : \Omega \to \mathcal{H}^k$ be $\mathcal{F}_0$-measurable and let $z = z(t)$, $t \in [0,T]$ be an $\mathcal{H}^k$-valued continuous process satisfying the following integral equation

$$z(t) = S_t z_0 + \int_0^t S_{t-s}\big(0, f(s)\big)\, ds + \int_0^t S_{t-s}\big(0, g(s)\big)\, dW(s),\ t \in [0,T]. \tag{3.12}$$

Given $\lambda \geq 0$ and $x \in \mathbb{R}^d$, we define the energy function $e : [0,T] \times \mathcal{H}^k \to \mathbb{R}^+$ by, for $z = (u,v) \in \mathcal{H}^k$,

$$\mathbf{e}(t,z) = \frac{1}{2} \left\{ \lambda |u|^2_{L^2(B(x,T-t))} + \sum_{l=0}^{k} \left[|D_x^{l+1} u|^2_{L^2(B(x,T-t))} + |D_x^l v|^2_{L^2(B(x,T-t))} \right] \right\}. \tag{3.13}$$

Assume that $L : [0,\infty) \to \mathbb{R}$ is a non-decreasing C^2-class function and define the second energy function $\mathbf{E} : [0,T] \times \mathcal{H}^k \to \mathbb{R}$, by

$$\mathbf{E}(t,z) = L(\mathbf{e}(t,z)),\ z = (u,v) \in \mathcal{H}^k.$$

and let a function $V : [0,T] \times \mathcal{H}^k \to \mathbb{R}$ be defined by, for $(t,z) \in \mathbb{R} \times \mathcal{H}^k$,

$$\begin{aligned} V(t,z) &= L'(\mathbf{e}(t,z)) \big[\lambda \langle u,v\rangle_{L^2(B(x,T-t))} + \langle v, f(t)\rangle_{H^k(B(x,T-t))}\big] \\ &+ \frac{1}{2} L'(\mathbf{e}(t,z)) \|g(t)\|^2_{\mathcal{T}_2(H_\mu, H^k(B(x,T-t)))} + \frac{1}{2} L''(\mathbf{e}(t,z)) |g(t)^* v|^2_{H_\mu}. \end{aligned}$$

Then $\mathbf{E} : [0,T] \times \mathcal{H}^k \to \mathbb{R}$ is continuous function and for every $t \in [0,T]$,

$$\begin{aligned} \mathbf{E}(t,z(t)) &\leq \mathbf{E}(0,z_0) + \int_0^t V(r,z(r))\,dr \\ &+ \int_0^t L'(\mathbf{e}(r,z(r))) \langle v(r), g(r)\,dW(r)\rangle_{H^k(B(x,T-r))}. \end{aligned} \tag{3.14}$$

Remark 3.3 - It can be shown that if an $\mathcal{H}^1$-valued process $z = (u,v)$ satisfies the integral equation (3.12) then the $H^2(\mathbb{R},\mathbb{R}^d)$-valued process u satisfies for all $t \geq 0$ in $L^2(\mathbb{R},\mathbb{R}^d)$, $\mathbb{P}$ almost surely,

$$\partial_t u(t) = v_0 + \int_0^t \big[\partial_{xx} u(s) + f(s)\big]\,ds + \int_0^t g(s)\,dW(s). \tag{3.15}$$

And vice versa, if the $H^2(\mathbb{R},\mathbb{R}^d)$-valued process u satisfies (3.15), then the $\mathcal{H}^1$-valued process $z = (u, \partial_t u)$ satisfies the integral equation (3.12). In particular, if Assumptions 3.2 are satisfied and process $u : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to M$ satisfies the five conditions (i)-(v) from Definition 3.1 then it satisfies condition (vii) from Definition 3.2 iff a process $z = (u, \partial_t u)$ satisfies the following condition:

(viii) For all $t \geq 0$ and $R > 0$ the following equality holds in $L^2((-R,R),\mathbb{R}^d)$, $\mathbb{P}$ almost surely,

$$\begin{aligned} z(t) &= S_t z_0 + \int_0^t S_{t-s}\big(0, -A_{u(s)}(\partial_x u(s), \partial_x u(s)) + A_{u(s)}(\partial_s u(s), \partial_s u(s))\big)\,ds \\ &+ \int_0^t S_{t-s}\big(0, Y_{u(s)}(\partial_s u(s), \partial_x u(s))\big)\,dW(s). \end{aligned} \tag{3.16}$$

We will also need the following generalization of Itô Lemma, see [3, Lemma 6.5].

Lemma 3.1. *Let U and K be separable Hilbert spaces, and let f and g be progressively measurable processes with values in K and $\mathcal{T}_2(U,K)$ respectively, such that*

$$\int_0^T \left\{ |f(s)|_K + \|g(s)\|^2_{\mathcal{T}_2(U,K)} \right\} ds < \infty \quad \text{almost surely.}$$

For some K-valued $\mathcal{F}_0$-measurable random variable ξ define a process z by

$$z(t) = S_t\xi + \int_0^t S_{t-s}f(s)\,ds + \int_0^t S_{t-s}g(s)\,dW(s), \qquad t \in [0,T],$$

where W is a cylindrical Wiener process on U, and $(S_t)_{t\geq 0}$ is a C_0-semigroup on K with an infinitesimal generator A. Let V be another separable Hilbert space and let $(T_t)_{t\geq 0}$ be a C_0-semigroup on V with an infinitesimal generator B. Suppose that $Q : K \to V$ is a C^2-smooth function such that $Q[D(A)] \subseteq D(B)$ and there exists a continuous function $F : K \to V$ such that

$$Q'(z)Az = BQ(z) + F(z), \qquad z \in D(A). \tag{3.17}$$

Then, for all $t \geq 0$,

$$\begin{aligned} Q(z(t)) &= T_tQ(\xi) + \int_0^t T_{t-s}Q'(z(s))g(s)\,dW(s) \\ &+ \int_0^t T_{t-s}\Big[Q'(z(s))f(s) + F(z(s)) + \frac{1}{2}\mathrm{tr}_K Q''(z(s)) \circ (g(s),g(s))\Big]\,ds. \end{aligned}$$

4. Elements of proofs

The basic idea of the proof of the main result comes from [18] and [1]. The nonlinearities A and Y in the equation (3.8) are extended from their domains (products of tangent bundles) to the ambient space, and thus we obtain a classical SPDE in a Euclidean space for which existence of global solutions is known. However our proof of the existence of the manifold valued solutions requires, that from the many extensions that can be constructed, we choose those which satisfy certain "symmetry" properties.

Let us denote by TM and NM the tangent and the normal bundle respectively, and denote by $\mathcal{E}$ the exponential function $T\mathbb{R}^d \ni (p,\xi) \mapsto p+\xi \in \mathbb{R}^d$ relative to the riemannian manifold $\mathbb{R}^d$ equipped with the standard Euclidean metric. The following result about tubular neighbourhood of M can be found in [26], see Proposition 7.26, p. 200.

Proposition 4.1. *There exists an $\mathbb{R}^d$-open neighbourhood O of M and an NM-open neighbourhood V around the set $\{(p,0) \in NM : p \in M\}$ such that the restriction of the exponential map $\mathcal{E}|_V : V \to O$ is a diffeomorphism. Moreover, V can be chosen in such a way that $(p,t\xi) \in V$ whenever $t \in [-1,1]$ and $(p,\xi) \in V$.*

Remark 4.1 - In what follows, we will denote the diffeomorphism $\mathcal{E}|_V : V \to O$ by $\mathcal{E}$, unless there is a danger of ambiguity.
Denote by $i : NM \to NM$ the diffeomorphism $(p,\xi) \mapsto (p,-\xi)$ and define

$$h = \mathcal{E} \circ i \circ \mathcal{E}^{-1} : O \to O. \tag{4.1}$$

The function h defined above is an involution on the normal neighbourhood O of M and corresponds to multiplication by -1 in the fibers, having precisely M for its fixed point set. The identification of the manifold M as a fixed point set of a smooth function enables to prove that solutions of heat equations with initial values on the manifold remain thereon, see [18] for deterministic heat equations in manifolds and [1] for stochastic heat equations in manifolds. Employing a partition of unity argument we may assume that $h : \mathbb{R}^d \to \mathbb{R}^d$ is such that properties (1)-(5) of Corollary 4.1 are valid on O. Therefore, it is without loss of generality to assume that the function h is defined on the whole $\mathbb{R}^d$.

Corollary 4.1. *The function h has the following properties: (i) $h : O \to O$ is a diffeomorphism, (ii) $h(h(q)) = q$ for every $q \in O$, (iii) if $q \in O$, then $h(q) = q$ if and only if $q \in M$, (iv) if $p \in M$, then $h'(p)\xi = \xi$, provided $\xi \in T_pM$ and $h'(p)\xi = -\xi$, provided $\xi \in N_pM$.*

Next we define, for $q \in \mathbb{R}^d$ and $a, b \in \mathbb{R}^d$,

$$B_q(a,b) = d_q^2 h(a,b), \quad \mathcal{A}_q(a,b) = \frac{1}{2} B_{h(q)}(h'(q)a, h'(q)b). \tag{4.2}$$

Let us recall that the second fundamental form A was introduced around the formula (2.3). We will be studying problem (3.7) with A replaced by $\mathcal{A}$. The following result is essential for our paper.

Proposition 4.2. *If $p \in M$ and $q \in O$, then*

$$\mathcal{A}_p(\xi,\eta) = \frac{1}{2} B_p(a,b) = A_p(\xi,\eta), \ \xi,\eta \in T_pM, \tag{4.3}$$

$$\mathcal{A}_{h(q)}(h'(q)a, h'(q)b) = h'(q)\mathcal{A}_q(a,b) + B_q(a,b), \ a,b \in \mathbb{R}^d. \tag{4.4}$$

Proof – Let $p \in M$, $\xi \in T_pM$ and let X be a vector field on M defined around p such that $X(p) = \xi$, and let $\widetilde{X}$ be a smooth $\mathbb{R}^d$-valued extension of X to aln $\mathbb{R}^d$-open neighbourhood V of p. Define next a smooth $\mathbb{R}^d$-valued function Y by $Y(q) = h'(q)\widetilde{X}(q)$, $q \in V$. Then by (iv) in Corollary 4.1, Y is an extension of X, i.e. $Y(q) = X(q) = \widetilde{X}(q)$, $q \in M \cap V$. Hence, for each $q \in M \cap V$ and $\eta \in T_qM$, $(d_qY)(\eta) = (d_q\widetilde{X})(\eta)$ and moreover by (2.3)

$$(d_pY)(\eta) = \nabla_\eta X + A_p(X(p),\eta) = \nabla_\eta X + A_p(\xi,\eta), \ \eta \in T_pM.$$

On the other hand, by (2.3) and Corollary 4.1, for $\eta \in T_pM$,

$$\begin{aligned}(d_pY)(\eta) &= B_p(\xi,\eta) + h'(p)(d_p\widetilde{X})(\eta) \\ &= B_p(\xi,\eta) + h'(p)\left[\nabla_\eta X + A_p(\xi,\eta)\right] = B_p(\xi,\eta) + \nabla_\eta X - A_p(\xi,\eta).\end{aligned}$$

Thus, by Corollary 4.1, for $\eta \in T_pM$,

$$\mathcal{A}_p(\xi,\eta) = \frac{1}{2} B_{h(p)}(h'(p)\xi, h'(p)\eta) = \frac{1}{2} B_p(\xi,\eta) = A_p(\xi,\eta).$$

For the second part, we have by definition and by Corollary 4.1 that

$$\mathcal{A}_{h(q)}(h'(q)a, h'(q)b) = \frac{1}{2} B_{h(h(q))}(h'(h(q))h'(q)a, h'(h(q))h'(q)b) = \frac{1}{2} B_q(a,b).$$

On the other hand, by the Leibnitz formula[2] applied to the equality $h \circ h = \mathrm{id}$ on $\mathcal{O}$, from (ii) in Corollary 4.1 we infer that

$$B_{h(q)}(h'(q)a, h'(q)b) = -h'(h(q))B_q(a,b).$$

Hence, since $h'(q)h'(h(q)) = \mathrm{id}$, we infer that

$$h'(q)\mathcal{A}_q(a,b) + B_q(a,b) = -\frac{1}{2}h'(q)h'(h(q))B_q(a,b) + B_q(a,b) = \frac{1}{2}B_q(a,b)$$

what completes the proof. □

To this end, let π_p, $p \in M$ be the orthogonal projection of $\mathbb{R}^d$ to T_pM and let us define $v_{ij}(p) = A_p(\pi_p e_i, \pi_p e_j)$ for $i,j \in \{1,\dots,n\}$ and extend the functions $v_{ij} = v_{ji}$ smoothly to the whole $\mathbb{R}^d$. The following result is just [3, Lemma 10.1].

Lemma 4.1. *Let $\gamma : (a,b) \to M$, where $(a,b) \subset \mathbb{R}$, be an H^1-smooth curve on M, and let X, Z be H^1-smooth vector fields along γ. Then, for a.a. $x \in (a,b)$,*

$$\begin{aligned} \langle \partial_x X, \partial_x[A_\gamma(Z,Z)]\rangle &= \sum_{i,j} \langle \partial_x X, \partial_k v_{ij}(\gamma)\partial_x\gamma^k Z^i Z^j\rangle \\ &- 2\sum_{i,j}\langle X, \partial_k v_{ij}(\gamma)\partial_x\gamma^k \partial_x Z^i Z^j\rangle. \end{aligned} \tag{4.5}$$

Now we will shortly recall the construction of extensions of vector fields on M to vector fields on O from [18], cf. [5]. To this end, let us define a new riemannian metric g on O by

$$g_q(a,b) = \langle a,b\rangle_{\mathbb{R}^d} + \langle h'(q)a, h'(q)b\rangle_{\mathbb{R}^d}, \qquad q \in O, \quad a,b \in \mathbb{R}^d. \tag{4.6}$$

Remark 4.2 - $h : (O,g) \to (O,g)$ is an isometric diffeomorphism.

If $q \in O$ then, by Proposition 4.1, there exists a unique $(p,\xi) \in V$ such that $q = \mathcal{E}(p,\xi)$. We will write $p(q) = p$ for this dependence. Moreover, also by Proposition 4.1, $\mathcal{E}(p,t\xi) \in O$ for $t \in [0,1]$. Hence we can define the curve

[2] If E, F, G are Banach spaces and $f : E \to F$, $g : F \to G$ are of twice differentiable at resp. $a \in E$ and $f(a)$, then $(d_a^2(g\circ f))(x_1,x_2) = (d^2_{f(a)}g)\big((d_af)x_1,(d_af)x_2\big) + (d_{f(a)}g)\big(d_a^2 f(x_1,x_2)\big)$, for all $x_1,x_2 \in E$.

$\gamma_q : [0,1] \ni t \mapsto \mathcal{E}(p, t\xi) \in O$. If $a \in \mathbb{R}^d$ and $(X(t))_{t\in[0,1]}$, $X(0) = a$ is the parallel translation of a along γ_q in (O, g) then we denote by $P_q a$ the endpoint vector $X(1)$.

Proposition 4.3. *[3, Proposition 3.9] $P : O \to \mathcal{L}^{\mathrm{isom}}(\mathbb{R}^n, \mathbb{R}^n)$ is a smooth function. Moreover, $P_q = I$ for $q \in M$ and*

$$h'(q)P_q = P_{h(q)}h'(p(q)), \qquad q \in O.$$

Due to this setting, it is possible to extend conveniently various mappings defined on the manifold M to its neighbourhood O, cf. [5].

For example, we have the following technical result whose proof can be found in [3].

Proposition 4.4. *If $R : (TM)^k \to TM$ is a vector bundle homomorphism, then there exists an extension $\widetilde{R}$ of R the whole space $\mathbb{R}^d$ which satisfies, for all $q \in O$ and $a_1, \dots, a_k \in \mathbb{R}^d$*

$$\widetilde{R}_q(a_1, \dots, a_k) = P_q R_{p(q)}(\pi_{p(q)} P_q^{-1} a_1, \dots, \pi_{p(q)} P_q^{-1} a_k) \in \mathbb{R}^d, \tag{4.7}$$

$$\widetilde{R}_{h(q)}(h'(q)a_1, \dots, h'(q)a_k) = h'(q)\widetilde{R}_q(a_1, \dots, a_k), \tag{4.8}$$

where for $p \in M$, π_p is the orthogonal projection from $\mathbb{R}^d$ to T_pM. Moreover, if R is smooth then so is $\widetilde{R}$.

5. Approximated Non-linearities

5.1. Existence of approximate solutions

Since we expect that the solutions of the equation (3.8) live on the manifold M, we cannot expect them to belong to the Hilbert space $H^2(\mathbb{R}) \oplus H^1(\mathbb{R})$, and, accordingly with the PDE theory, they will take values rather in the Fréchet space $H^2_{\mathrm{loc}}(\mathbb{R}) \oplus H^1_{\mathrm{loc}}(\mathbb{R})$. To overcome the problem with Bochner and stochastic integration which is not available in Fréchet spaces, we localize the problem by a series of non-linear wave equations.

Let us fix $T > 0$, $r > 2T$ and $k \in \mathbb{N}$. Let $\varphi = \varphi_r : \mathbb{R} \to \mathbb{R}$ be a smooth compactly supported function such that $\varphi = 1$ on $(-r, r)$. Let us recall, see e.g.

[22] that there exists a linear bounded operator $E^1 : H^1(-1,1) \to H^1(\mathbb{R})$ such that *(i)* $E^k f = f$ almost everywhere on $(-1,1)$ whenever $f \in H^1(-1,1)$, *(ii)* $E^1 f$ vanishes outside of $(-2,2)$ whenever $f \in H^1(-1,1)$, *(iii)* $E^1 f \in C^1(\mathbb{R})$, if $f \in C^1([-1,1])$ and *(iv)* there exists a unique extension of E^1 to a bounded linear operator from $H^0(-1,1)$ to $H^0(\mathbb{R})$. Let, for $r > 0$ an operator $E^1_r : H^j(-r,r) \to H^j(\mathbb{R})$, $j = 0,1$ be defined following formula

$$(E^1_r f)(x) = \{E^1[f(r\cdot)]\}(\frac{x}{r}), \qquad x \in \mathbb{R},\ f \in H^j(-r,r),\ j = 0,1. \tag{5.1}$$

In a similar manner we define extension operators E^2_r.

Note that the tangent bundle $T\mathbb{R}^d$ is isomorphic to $\mathbb{R}^d \times \mathbb{R}^d$ and the cartesian product bundle $T\mathbb{R}^d \times T\mathbb{R}^d$ is isomorphic to $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$. Using the formula (4.7) and Proposition 4.4 we can find an extension $Y : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ of the original fiber bundle homomorphism $Y : TM \times TM \to TM$ such that

$$Y_{h(q)}(h'(q)a_1, h'(q)a_2) = h'(q)Y_q(a_1, a_2), \qquad q \in O, \quad a_1, a_2 \in \mathbb{R}^d. \tag{5.2}$$

Let us fix $r > T > 0$. We define, with the notational convention that $z = (u,v) \in \mathcal{H}$, maps $\mathbf{F}^r = (0, \Phi^r) : [0,T] \times \mathcal{H} \to \mathcal{H}$, $\mathbf{G}^r = (0, \Gamma^r) : [0,T] \times \mathcal{H} \to \mathcal{T}_2(H_\mu, \mathcal{H})$ and $\mathbf{Q}_r : \mathcal{H} \ni z \mapsto (\varphi \cdot h(u), \varphi \cdot h'(u)v) \in \mathcal{H}$ by the following formulae,

$$\Phi^r_t(z) = E^1_{r-t}[\mathcal{A}_u(v,v) - \mathcal{A}_u(u_x, u_x)],\ (\Gamma^r_t(z))\xi = (E^1_{r-t} Y_u(v, u_x))\xi,\ \xi \in H_\mu.$$

Next, we define a map $\mathbf{F}_{r,k} : [0,T] \times \mathcal{H} \to \mathcal{H}$ (and by analogous formula a map $\mathbf{G}_{r,k} : [0,T] \times \mathcal{H} \to \mathcal{T}_2(H_\mu, \mathcal{H})$) by

$$\mathbf{F}^{r,k}_t(z) \quad = \quad \begin{cases} \mathbf{F}^r_t(z), & \text{if} \quad |z|_{\mathcal{H}_{r-t}} \le k, \\ \left(2 - \frac{1}{k}|z|_{\mathcal{H}_{r-t}}\right) \mathbf{F}^r_t(z), & \text{if} \quad k \le |z|_{\mathcal{H}_{r-t}} \le 2k, \\ 0, & \text{if} \quad 2k \le |z|_{\mathcal{H}_{r-t}}. \end{cases}$$

We begin with the following result which follows directly from Corollary 4.1 parts (3) and (4) and the definitions of the functions Q_r and φ.

Lemma 5.1. *If $z = (u,v) \in \mathcal{H}$ is such that $u(s) \in M$ and $v(s) \in T_{u(s)}M$ for $|s| < r$, then $Q_r(z) = z$ on $(-r,r)$.*

Lemma 5.2. *Let $k \in \mathbb{N}$. Then the functions $\mathbf{F}^r$, $\mathbf{F}_{r,k}$, $\mathbf{G}^r$, $\mathbf{G}_{r,k}$ are continuous and there exists a constant $C_{r,k}$ such that for all $t \in [0,T]$ and $z, w \in \mathcal{H}$*

$$|\mathbf{F}^{r,k}_t(z) - \mathbf{F}^{r,k}_t(w)|_{\mathcal{H}} + \|\mathbf{G}^{r,k}_t(z) - \mathbf{G}^{r,k}_t(w)\|_{\mathcal{T}_2(H_\mu, \mathcal{H})} \le C_{r,k}|z - w|_{\mathcal{H}_{r-t}}.$$

The following result is a direct consequence of the above Lemma 5.2 and e.g. Theorem 7.4 in [9]. We fix initial data u_0, V_0 as in Assumptions 3.2 and for each $r > 0$ define $\mathcal{H}$-valued $\mathcal{F}_0$-measurable random variable ξ_r by $\xi_r := \big(E_r^2(u_0), E_r^1(v_0)\big)$.

Corollary 5.1. *There exists a unique $\mathcal{H}$-valued continuous process $z_{r,k} = (u_{r,k}, v_{r,k})$ satisfying $\mathbb{E}\int_0^T |z_{r,k}(t)|^2_{\mathcal{H}}\, ds < \infty$ and such that for all $t \in [0,T]$, $\mathbb{P}$-a.s.*

$$z_{r,k}(t) = S_t\xi_r + \int_0^t S_{t-s}\mathbf{F}_s^{r,k}(z_{r,k}(s))\, ds + \int_0^t S_{t-s}\mathbf{G}_s^{r,k}(z_{r,k}(s))\, dW(s). \tag{5.3}$$

We will apply Lemma 3.1 to the process and the function $z_{r,k}$ and the function $\mathbf{Q}_r$. Thus we need to check that the assumptions are satisfied. We begin with the question of regularity of $\mathbf{Q}_r$.

From the Sobolev embedding $H^1(\mathbb{R}) \subseteq C_b(\mathbb{R})$ in conjunction with Assumption (3.1) we get the following result.

Lemma 5.3. *The map $\mathbf{Q}_r$ is of C^2-class and, with $z = (u,v), w, \theta \in \mathcal{H}$, it satisfies*

$$\begin{aligned}
\mathbf{Q}_r'(z)w &= (\varphi\cdot h'(u)w^1, \varphi\cdot[h''(u)(v,w^1) + h'(u)w^2]),\\
\mathbf{Q}_r''(z)(w,\theta) &= \Big(\varphi\cdot h''(u)(w^1,\theta^1), \varphi\cdot[h'''(u)(v,w^1,\theta^1)\\
&\qquad + h''(u)(w^1,\theta^2) + h''(u)(w^2,\theta^1)]\Big).
\end{aligned}$$

In particular, for all $(u,v) \in \mathcal{H}$ and $w \in H$, we have

$$\mathbf{Q}_r'(u,v)(0,w) = (0, \varphi\cdot h'(u)w), \tag{5.4}$$

$$\mathbf{Q}_r''(u,v)\big((0,w),(0,w)\big) = 0 \tag{5.5}$$

Moreover, $\mathbf{Q}_r'(z)A(z) = A\mathbf{Q}_r(z) + \mathbf{L}_r(z)$, $z \in \mathcal{H}$, where, with B being the map defined in (4.2), $\mathbf{L}_r : \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathbf{L}_r(z) = \big(0, \varphi B_u(v,v) - \varphi B_u(u_x,u_x) - \varphi_{xx}\cdot h(u) + 2\varphi_x\cdot h'(u)u_x\big).$$

Define now the following two auxiliary functions $\widetilde{\mathbf{F}}^{r,k} = (0, \tilde\Phi^{r,k}) : [0,T]\times \mathcal{H} \ni (t,z) \mapsto \mathbf{Q}_r'(z)\big(\mathbf{F}_t^{r,k}(z)\big) + \mathbf{L}_r(z) \in \mathcal{H}$ and $\widetilde{\mathbf{G}}^{r,k} = (0, \tilde\Gamma^{r,k}) : [0,T]\times\mathcal{H} \ni$

$(t,z) \mapsto \mathbf{Q}_r'(z) \circ \mathbf{G}_t^{r,k}(z) \in \mathcal{T}_2(H_\mu, \mathcal{H})$. In particular, if $r > t$ and $|x| < r - t$, then

$$\begin{aligned} \tilde{\Phi}_t^{r,k}(z) &= h'(u)(\mathbf{F}_t^{r,k})^2(z) + B_u(v,v) - B_u(u_x,u_x), && (5.6)\\ \tilde{\Gamma}_t^{r,k}(z) &= h'(u)(\Gamma_t^{r,k}(z)). && (5.7) \end{aligned}$$

This leads to the following result that rigorously expresses the fact that $\widetilde{\mathbf{F}}^{r,k}$, resp. $\widetilde{\mathbf{G}}^{r,k}$ is the push forward by Q of $\mathbf{F}^{r,k}$, resp. $\mathbf{G}^{r,k}$, provided $|x| < r - t$.

Proposition 5.1. *If $r > t$, then on $(-(r-t), r-t)$,*

$$\widetilde{\boldsymbol{F}}_t^{r,k}(z) = \mathbf{F}_t^{r,k}(Q_r(z)), \quad \widetilde{\boldsymbol{G}}_t^{r,k}(z) = \mathbf{G}_t^{r,k}(Q_r(z)). \tag{5.8}$$

Proof of Proposition 5.1– We begin with the second identity. In view of (5.4) it is enough to consider the second coordinates. By the invariance property of the vector field Y, see (5.2) and identities (5.4) and (5.7) we have,

$$\begin{aligned} \tilde{\Gamma}_t^{r,k}(z) &= h'(u) \circ \Gamma_t^{r,k}(z) = h'(u) \circ Y_u(v,u_x) = Y_{h(u)}(h'(v), h'(u_x)) \\ &= Y_{h(u)}(h'(v), [h(u)]_x)) = Y_{h(u)}(Q_r(z)) = \Gamma_t^{r,k}(Q_r(z)), \end{aligned}$$

what proves the first part (5.8). To prove the second we argue analogously but using (4.4) instead of (5.2) we have

$$\begin{aligned} \tilde{\Phi}_t^{r,k}(z) &= h'(u)\Phi_t^{r,k}(z) + B_u(v,v) - B_u(u_x,u_x) \\ &= h'(u)\mathcal{A}_u(v,v) - h'(u)\mathcal{A}_(u_x,u_x) + B_u(v,v) - B_u(u_x,u_x) \\ &= h'(u)\mathcal{A}_u(v,v) + B_u(v,v) - \Big[h'(u)\mathcal{A}_u(u_x,u_x) + B_u(u_x,u_x)\Big] \\ &= \mathcal{A}_{h(u)}(h'(u)v, h'(u)v) - \mathcal{A}_{h(u)}(h'(u)u_x, h'(u)u_x) \\ &= \mathcal{A}_{h(u)}(h'(u)v, h'(u)v) - \mathcal{A}_{h(u)}(h \circ u)_x, (h \circ u)_x) = \Gamma^{r,k}(t, Q_r(z)). \end{aligned}$$

□

Let us also observe that it follows from Lemma 5.3 that the assumptions of Lemma 3.1 are satisfied with the linear operator B being equal to A and the trace term equal to 0. Thus we have the following fundamental result in which the process $\widetilde{z}_{r,k}$ is defined by the following formuala

$$\widetilde{z}_{r,k} = \mathbf{Q}_r \circ z_{r,k}.$$

Corollary 5.2. *Let $z_{r,k}$ be the solution to* (5.3) *as in Corollary 5.1. Then for all $t \in [0,T]$, $\mathbb{P}$-a.s.,*

(5.9)

$$\widetilde{z}_{r,k}(t) = S_t \mathbf{Q}_r(\xi_r) + \int_0^t S_{t-s} \widetilde{\boldsymbol{F}}_s^{r,k}(z_{r,k}(s))\, ds + \int_0^t S_{t-s} \widetilde{\boldsymbol{G}}_s^{r,k}(z_{r,k}(s))\, dW(s).$$

5.2. Approximate solutions stay on the manifold

Let $z_{r,k} = (u_{r,k}, v_{r,k})$ be the solution to problem (5.3) and let us define, for each $k \in \mathbb{N}$, the following four functions from Ω to $[0,\infty]$.

$$\begin{aligned}
\tau_k^1 &= \inf\{t \in [0,T] : |z_{r,k}(t)|_{\mathcal{H}_{r-t}} > k\},\\
\tau_k^2 &= \inf\{t \in [0,T] : |\widetilde{z}_{r,k}(t)|_{\mathcal{H}_{r-t}} > k\},\\
\tau_k^3 &= \inf\{t \in [0,T] : \exists x \in [-(r-t), r-t] :\ u_{r,k}(t,x) \notin O\},\\
\tau_k &= \tau_k^1 \wedge \tau_k^2 \wedge \tau_k^3.
\end{aligned}$$

Lemma 5.4. *Each of the functions τ_k^j, $j = 1,2,3$, $j \in \mathbb{N}$, is a stopping time.*

PROOF – The process $z_{r,k}$ is continuous $\mathcal{H}_{r-t}$-valued and so the process $[0,T] \ni t \mapsto |z_{r,k}(t)|_{\mathcal{H}_{r-t}} \in \mathbb{R}_-$ is also continuous. Since τ_k^1 is the first exit time of the latter from the closed set $[0,k]$ and the filtration $\mathbb{F}$ is right-continuous, by Proposition 2.1.6 in [11], we infer that it is a stopping time. The same argument applies to τ_k^2.

Finally, let us observe that since $u_{r,k}$ is a jointly continuous process in (t,x), so τ_k^3 is equal to the first exit time of the continuous $\mathbb{F}$-adapted process B from the closed set $\{0\}$, where

$$B(t) = \inf\{\mathrm{dist}(u_{r,k}(t,x), \mathbb{R}^d \setminus O) : |x| \le r-t\},\ t \in [0,T].$$

□

For each $k \in \mathbb{N}$ we define an auxiliary process a_k by

$$\begin{aligned}
a_k(t) &= S_t \xi_r + \int_0^t S_{t-s} 1_{[0,\tau_k)}(s) \mathbf{F}_s^{r,k}(z_{r,k}(s))\, ds\\
&+ \int_0^t S_{t-s} 1_{[0,\tau_k)}(s) \mathbf{G}_s^{r,k}(z_{r,k}(s))\, dW(s),\ t \in [0,T].
\end{aligned}$$

Similarly, by replacing resp. ξ_r, $\mathbf{F}^{r,k}$ and $\mathbf{G}$ by resp. $\mathbf{Q}_r(\xi_r)$, $\widetilde{\mathbf{F}}^{r,k}$ and $\widetilde{\mathbf{G}}^{r,k}$ we define another auxiliary process $\widetilde{a}_k$.

Proposition 5.2. *The process a_k, $\widetilde{a}_k$, $z_{r,k}$ and $\widetilde{z}_{r,k}$ coincide on $[0,\tau_k)$ almost surely. In particular, $u_{r,k}(t,x)\in M$ for $|x|\le r-t$ and $t\in[0,\tau_k]$ almost surely. Consequently, $\tau_k=\tau_k^1=\tau_k^2\le\tau_k^3$.*

Proof – As in [19], since (S_t) is a C_0-group and so $S_{t-s}=S_t\circ S_{-s}$, the convolution integrals can be simply transformed into Itô integrals. Hence, by equalities (5.9) and (5.3) we infer that $a_k=z_{r,k}$ and $\widetilde{a}_k=\widetilde{z}_{r,k}$ on $[0,\tau_k)$. Now by Proposition 5.1 we infer that for all $s\in[0,T]$, $x\in[-(r-s),r-s]$, $\mathbb{P}$-a.s.

$$\begin{aligned} 1_{[0,\tau_k)}(s)[\widetilde{\mathbf{F}}_s^{r,k}(z_{r,k}(s))](x) &= 1_{[0,\tau_k)}(s)[\mathbf{F}_s^{r,k}(\widetilde{z}_{r,k}(s))](x) \\ 1_{[0,\tau_k)}(s)[\widetilde{\mathbf{G}}_s^{r,k}(z_{r,k}(s))e](x) &= 1_{[0,\tau_k)}(s)[\mathbf{G}_s^{r,k}(\widetilde{z}_{r,k}(s))e](x), \quad e\in H_\mu. \end{aligned}$$

Therefore, if $p(t)=|a_k(t)-\widetilde{a}_k(t)|^2_{\mathcal{H}_{r-t}}$, $t\in[0,T]$, then the stopped (at τ_k) process $s\mapsto p(s\wedge\tau_k)$, $s\in[0,T]$, is continuous and uniformly bounded. Note that since $\xi_r=\mathbf{Q}_r(\xi_r)$ on $(-r,r)$, $p(0)=0$. Moreover, by Proposition 3.1 and Lemma 5.2, we can find a continuous local martingale I with $I(0)=0$ such that for all $k\in\mathbb{N}$,

$$\begin{aligned} p(t\wedge\tau_k) &\le 2\int_0^t p(s\wedge\tau_k)\,ds+2C\int_0^t 1_{[0,\tau_k)}(s)|z_{r,k}(s)-\widetilde{z}_{r,k}(s)|^2_{\mathcal{H}_{r-s}}\,ds \\ &+ I(t\wedge\tau_k)\le 2(C+1)\int_0^t p(s\wedge\tau_k)\,ds+I(t\wedge\tau_k),\ t\in[0,T]. \end{aligned}$$

Let $(\sigma_j)_{j\in\mathbb{N}}$ be a sequence of stopping times that localizes I. Then for all $j,k\in\mathbb{N}$

$$p(t\wedge\tau_k\wedge\sigma_j)\le 2(C+1)\int_0^t p(s\wedge\tau_k\wedge\sigma_j)\,ds+I(t\wedge\tau_k\wedge\sigma_j)$$

and so by taking the expectation and then applying the the Gronwall lemma, we infer that $p=0$ on $[0,\tau_k\wedge\sigma_j]$ almost surely. By taking $j\to\infty$ limit we arrive at a conclusion that $p=0$ on $[0,\tau_k]$ almost surely. In other words, $a_k=\widetilde{a}_k$ on $[0,\tau_k]$ $\mathbb{P}$-a.s. and hence $\mathbb{P}$-a.s. $z_k=\widetilde{z}_k$ on $[0,\tau_k]$ as well. Consequently, $\mathbb{P}$-a.s. $u_{r,k}(t,x)\in O$ and $u_{r,k}(t,x)=h(u_{r,k}(t,x))$ for $|x|\le r-t$ and $t\in[0,\tau_k]$. Hence, by Corollary 4.1, $\mathbb{P}$-a.s. $u_{r,k}(t,x)\in M$ for $|x|\le r-t$ and $t\in[0,\tau_k]$. Consequently, $\tau_k\le\tau_k^3$ and so $\tau_k=\tau_k^1\wedge\tau_k^2$. Finally, since $p=0$ on $[0,\tau_k]$ we infer that $\tau_k^1=\tau_k^2$. □

5.3. The Approximate solutions extend each other

Proposition 5.3. *Let $k \in \mathbb{N}$. Then $z_{r,k+1}(t,x,\omega) = z_{r,k}(t,x,\omega)$ on $|x| \le r-t$, $t \in [0,\tau_k(\omega)]$, and $\tau_k(\omega) \le \tau_{k+1}(\omega)$ almost surely.*

PROOF – Define now a process p by $p(t) = |z_{r,k+1}(t) - z_{r,k}(t)|^2_{H^1(r-t)\oplus L^2(r-t)}$, $t \in [0,T]$ and apply Proposition 3.1. Since $p(0) = 0$ we can find continuous local martingale I satisfying $I(0) = 0$ such that for all $t \in [0,T]$, $\mathbb{P}$-a.s.

$$\begin{aligned} p(t) \quad &\le \quad \int_0^t 2p(s)\,ds + \int_0^t |1_{[0,\tau_{k+1})}\Phi_r(z_{r,k+1}) - 1_{[0,\tau_k)}(s)\Phi_r(z_{r,k})|^2_{L^2(r-s)}\,ds \\ &+ \quad \int_0^t |1_{[0,\tau_{k+1})}\Gamma_r(z_{r,k+1}) - 1_{[0,\tau_k)}\Gamma_r(z_{r,k})|^2_{\mathcal{T}_2(H_\mu,L^2(r-s))}\,ds + I(t). \end{aligned}$$

By the localization technique alreaday used in the proof of Proposition 5.2 we infer that $p = 0$ on $[0,\tau_{k+1}\wedge\tau_k]$. Hence by the definition of τ_k, $\tau_k \le \tau_{k+1}$. Indeed, if $|\xi_r|_{\mathcal{H}_r} > k+1$ then $\tau_{k+1} = \tau_k = 0$ and if if $k < |\xi_r|_{\mathcal{H}_r} \le k+1$ then $\tau_{k+1} > 0$ and $\tau_k = 0$. Thus, one can assume that $|\xi_r|_{\mathcal{H}_r} \le k$. If τ_{k+1} were smaller than τ_k then by the just proved property we would have $z_{r,k}(t) = z_{r,k+1}(t)$ for $t \in [0,\tau_{k+1}]$. Hence $|z_{r,k}(0)|_{\mathcal{H}_r} \le k$ and $|z_{r,k}(\tau_{k+1})|_{\mathcal{H}_{r-\tau_{k+1}}} \ge k+1$ and therefore we can find $\bar{t} \in [0,\tau_{k+1})$ such that $|z_{r,k}(\bar{t})|_{\mathcal{H}_{r-\bar{t}}} = k + \frac{1}{2}$. This implies that $\tau_k \le \bar{t}$ and this contradicts the assumption that $\tau_{k+1} < \tau_k$. The proof is complete. □

The stopping times (τ_k) are non-decreasing by Proposition 5.3, and so we can denote by τ the limit of (τ_k). Moreover, we can define a process $\tilde{z}_r(t,x)$, $t \in [0,\tau)$, $|x| \le r-t$ by $\tilde{z}_r(t,x,\omega) = z_{r,k}(t,x,\omega)$ provided k is so large that $\tau_k(\omega) = t$. Denote $\tilde{z}_r(t,x) = (\tilde{u}_r(t,x), \tilde{v}_r(t,x))$, $|x| \le r-t$. Note that $\tilde{z}_r(t,\cdot) \in \mathcal{H}_{r-t}$ and therefore, the following is a correct definition of an $\mathcal{H}$-valued process:

$$z_r(t) = \big(E^2_{r-t}\tilde{u}_r(t), E^1_{r-t}\tilde{v}_r(t)\big),\ t \in [0,\tau). \tag{5.10}$$

In the following two subsections we will show that $\tau = T$ $\mathbb{P}$-a.s. and then that the process $\tilde{z}_r = \big(\tilde{z}_r(t)\big)_{t\in[0,T)}$ is a solution to problem (3.16) with initial data ξ_r. Next, if $T < r_1 < r_2$, then $\xi_{r_1} = \xi_{r_2}$ on $[-r_1,r_1]$ and so by the uniqueness part of Theorem 3.2 (or rather by the same proof) it follows that $\tilde{z}_{r_1}(t,x) = \tilde{z}_{r_2}(t,x)$ for $t \in [0,T]$ and $|x| \le r_1 - t$. In this way can define a process $\tilde{z} = (\tilde{z}(t))_{t\in[0,T]}$ by $\tilde{z}(t,x,\omega) = \tilde{z}_r(t,x,\omega)$ provided r is so large that $r > |x| + t$. In the subsection 5.5 we will show that $\tilde{z}$ satisfies the condition (viii), in particular equality (3.16) in Remark 3.3.

5.4. No explosion for approximate solutions

Proposition 5.4. *$\tau = T$ almost surely.*

Proof – We first notice that due to the Chojnowska-Michalik Theorem, see [6] or Theorem 12 in [25], we have, for $t \in [0,T]$,

$$z_{r,k}(t) = \xi_r + \int_0^t \mathcal{G} z_{r,k}(s)\,ds + \int_0^t \mathbf{F}_s^{r,k}(z_{r,k}(s))\,ds + \int_0^t \mathbf{G}_s^{r,k}(z_{r,k}(s))\,dW(s). \tag{5.11}$$

In particular, with the integral converging in $\mathbb{H}^1(\mathbb{R})$, we have

$$u_{r,k}(t) = E_r^2(u_0) + \int_0^t v_{r,k}(s)\,ds,\ t \in [0,T].$$

Hence, as by the Sobolev embedding Theorem $\mathbb{H}^1(\mathbb{R}) \hookrightarrow C_b(\mathbb{R})$, we infer that $\partial_t u_{r,k}(t,x) = v_{r,k}(t,x)$ for all $t \in [0,T]$ and $x \in \mathbb{R}$, almost surely.

Next we define, for $t \geq 0$, $l(t) = |a_k(t)|^2_{\mathcal{H}^0_{r-t}}$, $q(t) = \log(1 + |a_k(t)|^2_{\mathcal{H}_{r-t}})$ and $\varphi(t) = \mathcal{A}_{u_{r,k}(t)}(v_{r,k}(t), v_{r,k}(t)) - \mathcal{A}_{u_{r,k}(t)}(\partial_x u_{r,k}(t), \partial_x u_{r,k}(t))$.

By applying Proposition 3.1 and Lemma 5.2 we can find continuous local martingales J_0, J_1 with $J_0(0) = J_1(0) = 0$ such that for all $t \in [0,T]$, almost surely,

$$\begin{aligned} l(t) \leq\ & l(0) + \int_0^t l(s)\,ds + \int_0^t 1_{[0,\tau_k]}(s)\langle v_{r,k}(s), \varphi(s)\rangle_{L^2(B_{r-s})}\,ds \\ & + \int_0^t 1_{[0,\tau_k]}(s)\|\mathbf{G}_s^{r,k}(z_{r,k}(s))\|^2_{\mathcal{T}_2(H_\mu, H^1(B_{r-s})\oplus L^2(B_{r-s}))}\,ds + J_0(t), \end{aligned} \tag{5.12}$$

$$\begin{aligned} q(t) \leq\ & q(0) + \int_0^t \frac{|a_k(s)|^2_{\mathcal{H}_{r-s}}}{1 + |a_k(s)|^2_{\mathcal{H}_{r-s}}}\,ds + \int_0^t 1_{[0,\tau_k]}(s)\frac{\langle v_{r,k}(s), \varphi(s)\rangle_{L^2(B_{r-s})}}{1 + |a_k(s)|^2_{\mathcal{H}_{r-s}}}\,ds \\ & + \int_0^t 1_{[0,\tau_k]}(s)\frac{\langle \partial_x v_{r,k}(s), \partial_x[\varphi(s)]\rangle_{L^2(B_{r-s})}}{1 + |a_k(s)|^2_{\mathcal{H}_{r-s}}}\,ds \\ & + \int_0^t 1_{[0,\tau_k]}(s)\frac{\|\mathbf{G}_s^{r,k}(z_{r,k}(s))\|^2_{\mathcal{T}_2(H_\mu,\mathcal{H})}}{1 + |a_k(s)|^2_{\mathcal{H}_{r-s}}}\,ds + J_1(t). \end{aligned} \tag{5.13}$$

Now, by Proposition 5.2, if $|x| \le r-s$ and $s \le \tau_k(\omega)$, then $u_{r,k}(s,x,\omega) \in M$ and so $u_{r,k}(s,x,\omega) \in M$ and $\partial_t u_{r,k}(s,x,\omega) = v_{r,k}(s,x,\omega) \in T_{u_{r,k}(s,x,\omega)}M$. Hence, by Proposition 4.2, almost surely on $|x| \le r-s$ and $s \le \tau_k(\omega)$,

$$\mathcal{A}_{u_{r,k}(s,x)}(v_{r,k}(s,x,\omega), v_{r,k}(s,x,\omega)) = A_{u_{r,k}(s,x)}(v_{r,k}(s,x), v_{r,k}(s,x)),$$
$$\mathcal{A}_{u_{r,k}(s,x)}(\partial_x u_{r,k}(s,x), \partial_x u_{r,k}(s,x)) = A_{u_{r,k}(s,x)}(\partial_x u_{r,k}(s,x), \partial_x u_{r,k}(s,x)).$$

Finally, since $v_{r,k} \in T_{u_{r,k}}M$ and $A_{u_{r,k}(s,x)}(\partial_x u_{r,k}(s,x), \partial_x u_{r,k}(s,x)) \in N_{u_{r,k}}M$, we infer that the integrands in the second integrals in (5.12) and (5.13) is equal to zero.

Also, by Lemma 5.2, for all $s \ge 0$,

$$\begin{aligned} 1_{[0,\tau_k)}(s)\|\mathbf{G}_s^{r,k}(z_{r,k}(s))\|^2_{\mathcal{T}_2(H_\mu,\mathcal{H}^0_{r-s})} &\le C_r 1_{[0,\tau_k)}(s)(1+l(s)) \\ 1_{[0,\tau_k)}(s)\|\mathbf{G}_s^{r,k}(z_{r,k}(s))\|^2_{\mathcal{T}_2(H_\mu,\mathcal{H})} &\le C_r 1_{[0,\tau_k)}(s)(1+l(s))(1+|a_k(s)|^2_{\mathcal{H}_{r-s}}). \end{aligned}$$

Therefore, inequality (5.12) becomes

$$l(t) \le l(0) + C_r \int_0^t (1+l(s))\,ds + J_0(t),\ t \ge 0.$$

In the same way as in subsection 5.2 by employing the localization argument, we can prove that for each $j \in \mathbb{N}$ there exists a constant $K_{r,j}$ such that with $B_j = \{\omega \in \Omega : |\xi_r(\omega)|_{\mathcal{H}_r} \le j\}$, one has

$$\mathbb{E}\, 1_{B_j}[1 + l(t \wedge \tau_k)] \le K_{r,j}, \qquad t \in [0,T], \quad j \in \mathbb{N}. \tag{5.14}$$

In order to deal with the third integral in (5.13) let us denote its integrand by $\zeta(s)$. By applying the Gagliardo-Nirenberg inequalities, see e.g. [13], we can show that $|\zeta(s)| \le C 1_{[0,\tau_k)}(s)(1+l(s))$, $s \ge 0$. Hence inequality (5.13) turns into

$$q(t) \le 1 + q(0) + C_r \int_0^t [1+l(s)]\,ds + J_1(t),\ t \ge 0.$$

As in the proof of inequality (5.14), by employing the localization argument and using (5.14), we can prove that for each $j \in \mathbb{N}$ there exists a constant $C_{r,j}$ such that

$$\mathbb{E}\, 1_{B_j} q(t \wedge \tau_k) \le C_{r,j}, \qquad t \in [0,T], \quad j \in \mathbb{N}. \tag{5.15}$$

Let us now fix $t \in [0,T)$. Then, since $1_{\{\tau_k \le t\}} |a_k(\tau_k)|_{\mathcal{H}_{r-\tau_k}} \ge k 1_{\{\tau_k \le t\}}$, we infer that

$$\log(1+k^2)\mathbb{P}\left(\{\tau_k \le t\} \cap B_j\right) \le \mathbb{E}\, 1_{B_j} q(t \wedge \tau_k) \le C_{r,j}. \tag{5.16}$$

Since $\tau_k \nearrow \tau$ as $k \to \infty$, from (5.16) we infer that for all $t \in [0,T)$, $j \in \mathbb{N}$, $\mathbb{P}\left(\{\tau \le t\} \cap B_j\right) = 0$ what in turn implies that $\tau = T$ almost surely. This completes the proof. □

5.5. Proof of the existence part of Theorem 3.2

This proof is continuations of our argument from the end of subsection 5.3.

Let us fix $R, T > 0$ and $r > T+R$. Since by (5.11), for $t \in [0,T]$, $\mathcal{H}^0$, $z_{r,k}(t \wedge \tau_k) = \xi_r + \int_0^{t\wedge\tau_k} \mathcal{G} z_{r,k}(s)\, ds + \int_0^{t\wedge\tau_k} \mathbf{F}_s^r(z_{r,k}(s))\, ds + \int_0^{t\wedge\tau_k} \mathbf{G}_s^r(z_{r,k}(s))\, dW(s)$. Restricting to the interval $(-R,R)$ and applying the natural projection from $\mathcal{H}^0$ to $\mathcal{H}_R^0$ the last equality becomes $\tilde{z}_r(t\wedge\tau_k) = \xi_r + \int_0^{t\wedge\tau_k} \mathcal{G}\tilde{z}_r(s)\, ds + \int_0^{t\wedge\tau_k} \mathbf{F}_s^r(\tilde{z}_r(s))\, ds + \int_0^{t\wedge\tau_k} \mathbf{G}_s^r(\tilde{z}_r(s))\, dW(s)$, in $\mathcal{H}_R^0$. Since $\tau_k \nearrow T$, we infer that in the $\mathcal{H}_R^0$ sense, for all $t \in [0,T)$,

$$\tilde{z}_r(t) = \xi_r + \int_0^t \mathcal{G}\tilde{z}_r(s)\, ds + \int_0^t \mathbf{F}_s^r(\tilde{z}_r(s))\, ds \, + \int_0^t \mathbf{G}_s^r(\tilde{z}_r(s))\, dW(s). \tag{5.17}$$

In particular, denoting $\tilde{z}_r = (\tilde{u}_r, \tilde{v}_r)$ and $\xi_r = (u_0^r, v_0^r)$, we have in $H^1(-R,R)$, $\tilde{u}_r(t) = u_0^r + \int_0^t \tilde{v}_r(s)\, ds$. Therefore, for $t \in [0,T]$, in the $L^2(-R,R)$ sense, $\tilde{v}_r(t) = v_0^r + \int_0^t \left[\partial_{xx}\tilde{u}_r(s) + A_{\tilde{u}_r(s)}(\tilde{v}_r(s), \tilde{v}_r(s)) - A_{\tilde{u}_r(s)}(\partial_x\tilde{u}_r(s), \partial_x\tilde{u}_r(s))\right] ds$ $+ \int_0^t Y_{\tilde{u}_r(s)}(\tilde{v}_r(s), \partial_x\tilde{u}_r(s))\, dW(s)$. In the formula above, we can already put in A because $\tilde{u}_r(t,x) = u_{r,k}(t,x) \in M$ for $|x| \le r-t$ and $t \in [0,T]$ by Propositions 5.2 and 5.4. Finally, we notice that in view of the definition of the process z, $z(t) = z_r(t)$ on $(-R,R)$ for $t \in [0,T]$. Therefore, from the last two equalities we infer that the process $z = (u,v)$ satisfies the condition (vii) in Definition 3.2. This concludes the proof of the existence part of Theorem 3.2. □

5.6. Proof of the uniqueness of Theorem 3.2

Let us fix $R > T > 0$ and consider processes u and $\bar{u}$ satisfying the conditions (•1)-(•6) listed in Theorem 3.2. Define an $\mathcal{H}$-valued continuous processes $Z = (u^R, v^R)$, $\bar{Z} = (\bar{u}^R, \bar{v}^R)$ by formula $Z(t,\omega) := \left(E_R^2 u(t,\omega), E_R^1 \partial_t u(t,\omega)\right)$, $t \in$

$[0,T]$ and analogously $\bar{Z}$. For $k\in\mathbb{N}$ let us define a stopping time $\sigma_k := \inf\{t\in [0,T] : \max\{|Z(t)|_{\mathcal{H}_{R-t}}, |\bar{Z}(t)|_{\mathcal{H}_{R-t}}\}\geq k\}$ and an $\mathcal{H}$-valued continuous process $\beta(t)=S_t(Z(0))+\int_0^t S_{t-s}1_{[0,\sigma_k)}(s)\mathbf{F}_s^{r,k}(Z(s))\,ds+\int_0^t S_{t-s}1_{[0,\sigma_k)}(s)\mathbf{G}_s^{r,k}(Z(s))\,ds$, $t\in[0,T]$. Then by Remark 3.3 the process u^R satisfies the equality (3.15) with $v_0 = v^R(0)$, $f(s) = 1_{[0,\sigma_k)}(s)\mathbf{F}_s^{r,k}(Z(s))$ and $g(s) = G_s^{r,k}(Z(s))$, for $s\in[0,T]$. By the defintions of σ_k, $\mathbf{F}^{r,k}$ and $\mathbf{G}^{r,k}$ we infer that for $s\in[0,T]$, $g(s) = 1_{[0,\sigma_k)}(s)Y_{u(s)}(\partial_s u(s),\partial_x u(s))$ and $f(s) = 1_{[0,\sigma_k)}(s)\mathcal{A}_{u(s)}(\partial_s u(s),\partial_s u(s))$ $-\mathcal{A}_{u(s)}(\partial_x U(s),\partial_x U(s))$. Hence, the process $U(t) := u(t)-u^R(t)$, solves the following homogenous *deterministic* wave equation

$$\partial_t U(t) = u_t(0) - v^R(0) + \int_0^t \partial_{xx} U(s)\,ds,\ t\in[0,\sigma_k]. \tag{5.18}$$

Note that $u_t(0,x)-v^R(0,x)=0$ for all $x\in[-R,R]$. Therefore, by the classical uniqueness result for deterministic wave equations (e.g. Chapter II, section 6 in [28]) we infer that $U(t,x,\omega)=0$ for $|x|\leq R-t$, $t\leq\sigma_k(\omega)$ almost surely. The same argument is valid for the process $\bar{u}$ and we can denote all object by addition of $\bar{}$, e.g. $\bar{\beta}$

If we define $q(t) = |\beta(t)-\bar{\beta}(t)|^2_{\mathcal{H}_{R-t}}$, by applying Proposition 3.1 we can find a continuous local martingale I is satisfying $I(0)=0$ such that

$$\begin{aligned} q(t\wedge\sigma_k) &\leq \int_0^{t\wedge\sigma_k} [2q(s) + |\mathbf{F}_s^{R,k}(Z(s)) - \mathbf{F}_s^{R,k}(\bar{Z}(s))|^2_{\mathcal{H}}]\,ds + I(t\wedge\sigma_k) \\ &+ \int_0^{t\wedge\sigma_k} \|\mathbf{G}_s^{R,k}(Z(s)) - \mathbf{G}_s^{R,k}(\bar{Z}(s))\|^2_{\mathcal{T}_2(H_\mu,\mathcal{H})}\,ds,\ t\in[0,T]. \end{aligned}$$

Lemma 5.2 implies that $q(t\wedge\sigma_k)\leq\int_0^{t\wedge\sigma_k} Cq(s)\,ds + I(t\wedge\sigma_k)$. By the localization argument we get $\mathbb{E}\,q(t\wedge\sigma_k)\leq\int_0^t C\mathbb{E}\,q(s\wedge\sigma_k)\,ds$, $t\in[0,T]$ and so by the Gronwall Lemma we infer that $q=0$ on $[0,\sigma_k)$. Hence, since $q(t)\geq 1_{[0,\sigma_k)}|u(t)-\bar{u}(t)|^2_{\mathbb{L}^2(-(R-t),R-t)}$. Passing with k to infinity, we obtain that $u(t,x,\omega)=\bar{u}(t,x,\omega)$ for $|x|\leq R-t$, $t\in[0,T]$ as claimed. □

5.7. Equivalence of two definitions - Proof of Theorem 3.1

Assume that u is an *extrinsic* solution. Let X be a vector field on M, $R>0$ and $\mathcal{O}=(-R,R)$. Then, by (2.4) we have, in the $H^1(\mathcal{O};\mathbb{R}^d)$ sense,

$$X(u(t)) = X(u_0) + \int_0^t \big(\nabla_{\partial_s u(s)} X + A_{u(s)}(X(u(s)),\partial_s u(s))\big)\,ds,\ t\geq 0. \tag{5.19}$$

Hence, by applying the Itô Lemma, see e.g. Theorem 4.17 in [9], to equalities (5.19)-(3.7) and a function $\varphi : \mathbb{H}^1(\mathcal{O}) \times \mathbb{L}^2(\mathcal{O}) \ni (u,v) \mapsto \langle u, v\rangle_{\mathbb{L}^2(\mathcal{O})} \in \mathbb{R}$ we infer that

$$\begin{aligned}
\langle \partial_t u(t), X(u(t))\rangle &= \langle v_0, X(u_0)\rangle + \int_0^t \langle X(u(s)), Y_{u(s)}(\partial_s u(s), \partial_x u(s))\, dW(s)\rangle \\
&+ \int_0^t \Big\{ \langle X(u(s)), \partial_{xx} u(s) - A_{u(s)}(\partial_x u(s), \partial_x u(s)) \\
&+ A_{u(s)}(\partial_s u(s), \partial_s u(s))\rangle + \langle \partial_s u(s), \nabla_{\partial_s u(s)} X\rangle \Big\}\, ds, \ t \geq 0.
\end{aligned} \tag{5.20}$$

Since $A_p(\cdot,\cdot) \perp T_pM$ for all $p \in M$, we infer that $\langle X(u(s)), A_{u(s)}(\partial_s u(s), \partial_s u(s))\rangle = 0$ for all $s \geq 0$. Hence, by applying identity (2.5) to (5.20) we find that u satisfies the equality (3.5), i.e. that u is an *intrinsic* solution.

To prove the converse implication let us assume u is an *intrinsic* solution. Let $Z_1, \dots, Z_k$ be a a finite system of vector fields on M such that (2.7) holds true. Recall that the $\mathbb{R}^d$ components of Z_i are being denoted by $Z_i^1, \cdots, Z_i^n$.

From equation (3.5) applied to vector fields $X_i^j = Z_i^j Z_i$, $i \leq k$, $j \leq n$, multiplied by vector e_j and summed over j, by applying (2.7) we obtain

$$\begin{aligned}
\partial_t u(t) &= v_0 + \int_0^t \mathbf{D}_x \partial_x u(s)\, ds + \int_0^t \sum_{j=1}^n \sum_{i=1}^k \langle \partial_s u(s), \nabla_{\partial_s u(s)} (Z_i^j Z_i)\rangle\, e_j\, ds \\
&+ \int_0^t Y_{u(s)}(\partial_s u(s), \partial_x u(s))\, dW(s).
\end{aligned} \tag{5.21}$$

But, by (2.8), $\sum_{j=1}^n \sum_{i=1}^k \langle \partial_s u(s), \nabla_{\partial_s u(s)}(Z_i^j Z_i)\rangle\, e_j = A_{u(s)}(\partial_s u(s), \partial_s u(s))$ and hence, from (5.21) we infer that u satisfies the equality (3.7), i.e. that u is an extrinsic solution to problem (3.1) with initial data (3.4).

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